

# Spinodal Inflation

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Out-of-equilibrium, non-perturbative, quantum effects significantly modify the standard picture of inflation in a wide class of models including new, natural, and hybrid inflation. We find that the quantum evolution of a single real inflaton field may be modeled by a classical theory of *two* homogeneous scalar fields. We briefly discuss the important observational consequences that are expected to result.

The advent of inflation [1,2] is one of the most significant advances in our understanding of cosmology and the early universe in the past twenty years. It not only provides a solution to a number of shortcomings of the standard model of cosmology, but also provides the favored mechanism for the production of the primordial density perturbations which have been observed in the Cosmic Microwave Background and from which the large scale structure of our universe formed.

While there are any number of models of inflation, as a paradigm it remains remarkably simple. Even the simplest of scalar field theory models, with appropriately chosen parameters and initial states, are successful in producing a universe consistent with present observation. Furthermore, it has been thought that the inflaton may be treated as a purely *classical* field evolving in a *classical* effective potential, with quantum mechanics only entering in the formation of primordial density perturbations from vacuum fluctuations [3], and in the uncertainty in the inflaton initial condition.

Here we show that quantum effects can have far further ranging effects than previously thought. Potentials with spinodal instabilities, i.e. potentials  $V(\Phi)$  for which  $V''(\Phi)$  changes sign, such as those used for spontaneous symmetry breaking, demonstrate a phenomenon which we call *spinodal inflation*.

In spinodal inflation, the out-of-equilibrium, non-perturbative, quantum physics significantly modifies the evolution of the inflaton. The result is that the evolution of a real inflaton field may in fact be accurately modeled not by one, but by *two* coupled classical fields. To do this, we need to use the closed time path formalism of non-equilibrium quantum field theory to study the time evolution of a real inflaton [4]. The spinodal instabilities give rise to non-perturbative behavior that we treat by means of the self-consistent Hartree approximation [5],

while the gravitational evolution is included by means of the semi-classical approximation [6].

Doing this we will see that the long-wavelength fluctuations assemble themselves into an effective homogeneous *classical* field with only perturbatively small quantum corrections remaining [7]. This effective field, together with the original zero momentum mode are the two fields that can be used to describe the quantum non-equilibrium evolution of the full system.

The resulting phenomenology is rich and complex. Observables depend not only on the parameters of the inflationary theory, but also on the initial conditions. Furthermore, observables may take on values in the complete quantum theory that are not allowed according to the simple classical analyses.

We work in a spatially flat Friedmann-Robertson-Walker universe with metric

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2, \quad (1)$$

and we take the inflaton to be a real scalar field with Lagrangian

$$L = \frac{1}{2} \nabla^\mu \Phi(x) \nabla_\mu \Phi(x) - V[\Phi(x)].$$

We will be interested in the case in which the potential is even in  $\Phi$  with a negative squared mass, such that there is a local maximum at  $\Phi = 0$ .

It is convenient to break up the field  $\Phi$  into its expectation value, defined within the closed time path formalism, and fluctuations about that value:

$$\Phi(\vec{x}, t) = \phi(t) + \psi(\vec{x}, t), \quad \phi(t) \equiv \langle \Phi(\vec{x}, t) \rangle.$$

By definition  $\langle \psi(\vec{x}, t) \rangle = 0$ , while  $\phi$  depends only on time as a consequence of space translation invariance.

By imposing the Hartree factorization, we arrive at the following equations of motion for the inflaton [8]:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle \psi^2 \rangle^n V^{(2n+1)}(\phi) = 0, \quad (2)$$

$$\left[ \frac{d^2}{dt^2} + 3\frac{\dot{a}}{a} \frac{d}{dt} + \frac{k^2}{a^2} + \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle \psi^2 \rangle^n V^{(2n+2)}(\phi) \right] f_k = 0, \quad (3)$$

where

$$V^{(n)} \equiv \frac{\delta^n V(\phi)}{\delta \phi^n}.$$

The two-point fluctuation  $\langle \psi^2 \rangle$  is determined from the mode functions  $f_k$ :

$$\langle \psi^2 \rangle = \int \frac{d^3 k}{2(2\pi)^3} |f_k|^2. \quad (4)$$

For  $a(t_0) = 1$ , the initial conditions on the mode functions are

$$f_k(t_0) = \frac{1}{\sqrt{\omega_k}}, \quad \dot{f}_k(t_0) = (-\dot{a}(t_0) - i\omega_k) f_k(t_0), \quad (5)$$

with

$$\omega_k^2 \equiv k^2 + \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle \psi^2(t_0) \rangle^n V^{(2n+2)}(\phi(t_0)) - \frac{R(t_0)}{6}.$$

$R(t_0)$  is the initial Ricci scalar. For  $k^2 < |V^{(2)}(\phi(t_0))|$  we modify  $\omega_k$  either by means of a quench or by explicit deformation so that the frequencies are real. The exact choice corresponds to different initial vacuum states and has little effect on results [7,9].

The gravitational dynamics are determined by the semi-classical Einstein equation [6]. For a minimally coupled inflaton we have:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3} \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \langle \dot{\psi}^2 \rangle + \frac{1}{2a^2} \langle (\vec{\nabla} \psi)^2 \rangle + \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle \psi^2 \rangle^n V^{(2n)}(\phi) \right], \quad (6)$$

where  $G_N$  is Newton's gravitational constant, and

$$\langle \dot{\psi}^2(t) \rangle \equiv \int \frac{d^3 k}{2(2\pi)^3} |\dot{f}_k|^2, \quad (7)$$

$$\left\langle \left( \vec{\nabla} \psi(t) \right)^2 \right\rangle \equiv \int \frac{d^3 k}{2(2\pi)^3} k^2 |f_k|^2. \quad (8)$$

In what follows, we assume that each of these integrals has been regulated either because the theory is a low energy effective theory with a definite cutoff or because the divergences have been absorbed into a renormalization of the parameters of the theory [7,10].

We now describe zero mode assembly [7]. We write the potential  $V(\Phi)$  as

$$V(\Phi) = K - \frac{1}{2} \mu^2 \Phi^2 + \frac{\lambda}{4!} \Phi^4 + \dots, \quad (9)$$

where the constant  $K$  is chosen such that the potential is zero in the true vacuum. Initially, the fluctuations  $\langle \psi^2 \rangle$  are small, so for  $\phi_{\text{initial}} \ll \mu/\sqrt{\lambda}$ , the mode functions evolve as

$$\ddot{f}_k + 3H_0 \dot{f}_k + \left( \frac{k^2}{a^2} - \mu^2 \right) f_k = 0,$$

where  $H_0^2 \simeq 8\pi G_N K/3$ . As a result, those modes whose physical wavelength is greater than the horizon scale with

$k/a < H_0$  grow exponentially; this is the spinodal instability. After a few  $e$ -folds of inflation, the integral (4) becomes dominated by long wavelength modes, which are the most unstable ones, so that we may replace the quantity  $\langle \psi^2(t) \rangle^{1/2}$  by an effectively *homogeneous* and classical zero mode  $\sigma(t)$ . Furthermore, the gravitational evolution may be written in terms of  $\sigma$  with the replacements  $\langle \dot{\psi}^2 \rangle \rightarrow \dot{\sigma}^2$  and  $\langle (\vec{\nabla} \psi)^2 \rangle / a^2 \rightarrow 0$ . Both of these replacements are justified once a few  $e$ -folds of inflation have passed [7].

Finally, we arrive at the following system of equations for  $\phi$  and  $\sigma$ :

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \sum_{n=0}^{\infty} \frac{1}{2^n n!} \sigma^{2n} V^{(2n+1)}(\phi) = 0, \quad (10)$$

$$\ddot{\sigma} + 3\frac{\dot{a}}{a}\dot{\sigma} + \sum_{n=0}^{\infty} \frac{1}{2^n n!} \sigma^{2n+1} V^{(2n+2)}(\phi) = 0, \quad (11)$$

while

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3} \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\sigma}^2 + \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{2^n n!} V^{(2n)}(\phi) \right]. \quad (12)$$

Corrections to these effective classical equations due to modes which have not yet crossed the horizon will be perturbatively small [7].

Examination of the equations (10) – (12) reveals an amazing result. After a few  $e$ -folds of inflation, the full quantum evolution of spinodal models of inflation for a real scalar inflaton may be modeled by a classical system of two homogeneous fields with potential

$$V(\phi, \sigma) = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \sigma^{2n} V^{(2n)}(\phi). \quad (13)$$

We emphasize that the two-field reassembled system arises from the Hartree *dynamics* obtained by solving eqs.(2, 3,6), together with the initial conditions in eq.(5).

There are several important notes to make regarding this result. First, the initial value of  $\sigma$  is not a free parameter; rather, it is determined dynamically through the assembly process. By completing the full quantum evolution of  $\langle \psi^2 \rangle$  until assembly is evident and then extrapolating back to the initial time, we can show numerically that the effective initial value of  $\sigma$  is  $\sigma(t_0) = \mathcal{F}(H_0)H_0/2\pi$ , where  $\mathcal{F}(H_0)$  is a number of order 1 [7]. Because the effective initial value of  $\sigma$  is fixed by the dynamics, there is a clear separation into two regimes, depending upon the initial value of  $\phi$ . The first is the classical regime for which  $\phi(t_0) \gg H_0/2\pi$ ; here the dynamics will follow the usual classical dynamics for the original potential  $V(\phi)$ . However, there exists a second regime with  $\phi(t_0) < H_0/2\pi$  for which the quantum evolution modeled by the  $\sigma$  field will have a significant influence on the dynamics.

Second, while the dynamics described by the potential (13) looks like that of a two field model of inflation, this is only true in terms of the homogeneous dynamics; since the  $\phi$  and  $\sigma$  fields are in fact just two aspects of the *same* scalar field, the computation of the primordial power spectrum in these theories somewhat more subtle.

Now we turn to an example. Consider a spontaneously broken  $\lambda\Phi^4$  theory with a potential

$$V(\Phi) = \frac{3m^4}{2\lambda} - \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4. \quad (14)$$

This potential vanishes at its minima at  $\Phi = \pm\sqrt{6m^2/\lambda}$ . Of particular importance to us is the location of the spinodal line separating the region of spinodal instability from stability. This runs from  $\Phi = -\sqrt{2m^2/\lambda}$  to  $\Phi = +\sqrt{2m^2/\lambda}$ . In the large  $N$  case treated in [7], the presence of Goldstone modes makes the spinodal line run along the *minima* of the potential.

If we evolve eqs.(2, 3, 6) in time we arrive at the results in fig.1.

The “Hartreeized” potential generated by the growth of quantum fluctuations is

$$V_{\text{hartree}}(\phi, \sigma) = V(\phi) - \frac{\lambda}{4}(\phi_{\text{spinodal}}^2 - \phi^2)\sigma^2 + \frac{\lambda}{8}\sigma^4, \quad (15)$$

where  $\phi_{\text{spinodal}} = \sqrt{2m^2/\lambda}$ .

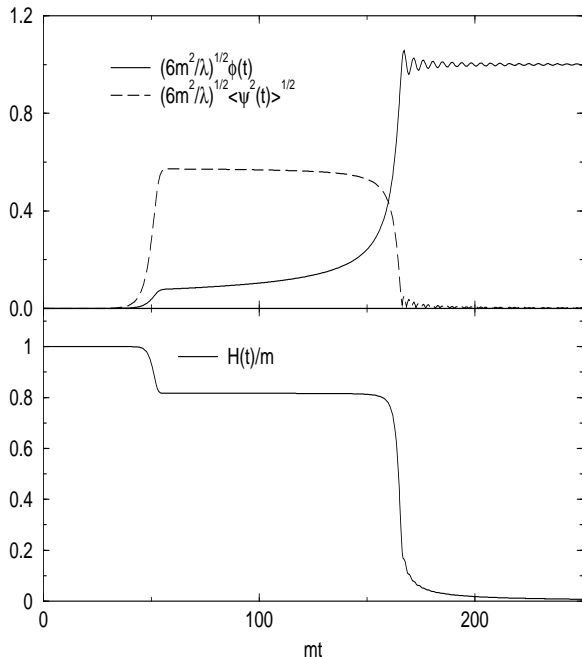


FIG. 1. Typical evolution of the zero mode  $\phi$ , the two-point fluctuation  $\langle\psi^2\rangle$ , and the expansion rate  $H \equiv \dot{a}/a$  in the  $\lambda\Phi^4$  theory. Here,  $\phi(t_0) = 0.2H_0/m$ ,  $H_0/m = 1$ , and  $\lambda = 8\pi^2 \times 10^{-14}$ .

We see that there are two inflationary stages. The first is driven by the vacuum energy at  $\phi \sim 0$ . The fluctuations then grow and end that period of inflation. However, once the fluctuations have grown large enough, the second period of inflation ensues. Comparing the value of the Hubble parameter  $H$  in both phases, we see that the system now behaves as if the order parameter was stuck near the *spinodal* value for the original potential  $V(\phi)$ ! In fact, the dynamics is found to obey the following sum rule during this second inflationary phase:

$$\langle\psi^2\rangle = \begin{cases} \phi_{\text{spinodal}}^2 - \phi^2 & \text{if } |\phi| \leq \phi_{\text{spinodal}} \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

In terms of the effective classical theory determined by the potential in Eq. (15), this sum rule corresponds to minimizing  $V_{\text{hartree}}(\phi, \sigma)$  with respect to  $\sigma$ :  $\partial_\sigma V_{\text{hartree}}(\sigma, \phi) = 0$ . Once  $\phi$  reaches the spinodal, the minimizing condition is that the fluctuations vanish and that  $\phi$  evolves to the minimum of the *tree-level* potential. In fact this is what the dynamics shows. Note that the sum rule is reminiscent of the behavior of the magnetization in a Heisenberg ferromagnet as a function of temperature.

The number of e-folds of the second phase of inflation is determined by the amount of time it takes  $\phi$  to reach  $\phi_{\text{spinodal}}$ ; in the extreme case that  $\phi$  is fixed at 0 this phase never ends.

It is a worthwhile exercise to compare this behavior to the behavior found in the large  $N$  approximation. There, zero mode assembly also occurred, but the effective zero mode just evolved along the *classical* potential to the tree-level minimum; there was no second inflationary phase. We would argue that we are seeing the same behavior in both cases, the only difference being that the existence of Goldstone modes in the large  $N$  case makes the spinodal run along the minima of the tree-level potential. Thus the reassembled zero mode *does* in fact go to the spinodal line. However, in large  $N$  that is just the line of minima and since the potential is chosen to be zero at the minima, there is no vacuum energy left to drive a second inflationary phase.

An alternative way to interpret what we are seeing here is that while field fluctuations grow, they are producing particles whose effect is to modify the background in which the field is evolving. These particles are produced copiously enough to survive the exponential redshift during the inflationary phase. The “potential” that the zero mode will follow is now one that must be computed in the presence of the bath of produced particles. Once the zero mode crosses the spinodal line, however, particle production ends and their effect is obliterated by redshifting, thus allowing the zero mode to find the minimum of the tree-level potential.

Finally, we address the issue of metric perturbations in spinodally unstable theories. There are two aspects

of the evolution which will result in departures from the conventional wisdom for this class of models. First is the two stage nature of the evolution which may result in a deviation from scale invariance and a kink in the power spectrum [11]. Second is the effective two field dynamics in which the zero mode slowly evolves to the spinodal line and triggers the end of inflation. This is reminiscent of hybrid inflation models and may have similar consequences, such as the production of a blue tilt to the power spectrum [12]. Any of these possibilities will have significant consequences for the reconstruction program [13]. A proper study using the gauge invariant formalism [3] is in progress [14].

In this work we have made heavy use of the Hartree approximation. It is reasonable to ask whether this approximation captures the essential physics of the situation. First we should note that, unlike the large  $N$  case, the Hartree approximation is a *truncation* of the theory that is uncontrolled in the sense that we do not have a way to systematically go beyond it. There is some hope that the 2PI formalism of Cornwall, Jackiw and Tomboulis [15], could be used to at least try to estimate the diagrams omitted in the Hartree truncation [14].

However, we should be heartened by the fact that the Hartree approximation *does* recognize the importance of the spinodal line. As shown by Weinberg and Wu [16], the existence of the spinodal line is correlated both with the non-convexity as well as with the imaginary part of the one-loop effective potential. This imaginary part corresponds to the decay rate of a state prepared so that the zero mode is localized near the top of the potential. Thus spinodal models are *inherently* dynamical and cannot be treated within the confines of the effective potential approximation. What the Hartree approximation provides us with is a way to deal with spinodal dynamics beyond perturbation theory.

The result is a dynamical evolution in which non-perturbative *quantum* fluctuations play a primary role in the evolution of the inflaton field, the gravitational background, and the production of primordial metric perturbations.

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